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**3 (Sem-5/CBCS) MAT HC 1 (N/O)**

**2023**

**MATHEMATICS**

(Honours Core)

**OPTION-A**

**(For New Syllabus)**

Paper : MAT-HC-5016

**(Complex Analysis)**

Full Marks : 60

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions :  $1 \times 7 = 7$

(a) Which point on the Riemann sphere represents  $\infty$  of the extended complex plane  $\mathbb{C} \cup \{\infty\}$ ?

(b) A set  $S \subseteq \mathbb{C}$  is closed if and only if  $S$  contains each of its \_\_\_\_\_ points.

*(Fill in the gap)*

*Contd.*

(c) Write down the polar form of the Cauchy-Riemann equations.

(d) The function  $f(z) = \sinh z$  is a periodic function with a period \_\_\_\_\_  
(Fill in the gap)

(e) Define a simple closed curve.

(f) Write down the value of the integral  $\int_C f(z) dz$ , where  $f(z) = ze^{-2}$  and  $C$  is the circle  $|z| = 1$ .

(g) Find  $\lim_{n \rightarrow \infty} z_n$ , where  $z_n = -1 + i \frac{(-1)^n}{n^2}$ .

2. Answer the following questions :  $2 \times 4 = 8$

(a) Let  $f(z) = i \frac{z}{2}$ ,  $|z| < 1$ . Show that

$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$ , using  $\varepsilon - \delta$  definition.

(b) Show that all the zeros of  $\sinh z$  in the complex plane lie on the imaginary axis.

(c) Evaluate the contour integral

$$\int_C \frac{dz}{z}, \text{ where } C \text{ is the semi circle } z = e^{i\theta}, 0 \leq \theta \leq \pi$$

(d) Using Cauchy's integral formula, evaluate

$$\int_C \frac{e^{2z}}{z^4} dz, \text{ where } C \text{ is the circle } |z| = 1.$$

3. Answer **any three** questions from the following :  $5 \times 3 = 15$

(a) Find all the fourth roots of  $-16$  and show that they lie at the vertices of a square inscribed in a circle centered at the origin.

(b) Suppose  $f(z) = u(x, y) + iv(x, y)$ ,  $(z = x + iy)$  and  $z_0 = x_0 + iy_0$ ,  $w_0 = u_0 + iv_0$ . Then prove the following :

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0,$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0, \text{ if and only}$$

$$\text{if } \lim_{z \rightarrow z_0} f(z) = w_0.$$

(c) (i) Show that the function  $f(z) = \operatorname{Re} z$  is nowhere differentiable.

(ii) Let  $T(z) = \frac{az+b}{cz+d}$ , where

$$ad - bc \neq 0.$$

Show that  $\lim_{z \rightarrow \infty} T(z) = \infty$  if  $c = 0$ .

3+2=5

(d) Let  $C$  be the arc of the circle  $|z|=2$  from  $z=2$  to  $z=2i$  that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$$

(e) State and prove fundamental theorem of algebra.

4. Answer **any three** questions from the following : 10×3=30

(a) (i) Show that  $\exp(z + \pi i) = -\exp(z)$  1

(ii) Show that  $\log(-1+i)^2 \neq 2\log(-1+i)$  2

(iii) Show that  $|\sin z|^2 = \sin^2 x + \sinh^2 y$  2

(iv) Show that a set  $S \subseteq \mathbb{C}$  is unbounded if and only if every neighbourhood of the point at infinity contains at least one point of  $S$ . 5

(b) (i) Suppose that  $f(z_0) = g(z_0) = 0$  and that  $f'(z_0)$ ,  $g'(z_0)$  exist with  $g'(z_0) \neq 0$ . Using the definition of derivative show that

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad 5$$

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

where  $|z| < \infty$ . 5

(c) State and prove Laurent's theorem.

(d) (i) Using definition of derivative, show that  $f(z) = |z|^2$  is nowhere differentiable except at  $z=0$ . 5

- (ii) Define singular points of a function. Determine singular points of the functions :

$$f(z) = \frac{2z+1}{z(z^2+1)} ;$$

$$g(z) = \frac{z^3+i}{z^2-3z+2} \quad 1+4=5$$

- (e) (i) Let  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D$ . Prove that the families of curves  $u(x, y) = c_1$ ,  $v(x, y) = c_2$  are orthogonal.

- (ii) Let  $C$  denote a contour of length  $L$  and suppose that a function  $f(z)$  is piecewise continuous on  $C$ . If  $M$  is a non-negative constant such that

$$|f(z)| \leq M \text{ for all } z \text{ in } C$$

then show that

$$\left| \int_C f(z) dz \right| \leq ML. \quad 5+5=10$$

- (f) (i) Prove that two non-zero complex numbers  $z_1$  and  $z_2$  have the same moduli if and only if  $z_1 = c_1 c_2$ ,  $z_2 = c_1 \bar{c}_2$ , for some complex numbers  $c_1, c_2$ . 4

- (ii) Show that mean value theorem of integral calculus of real analysis does not hold for complex valued functions  $w(t)$ . 3

- (iii) State Cauchy-Goursat theorem. 1

- (iv) Show that  $\lim_{z \rightarrow \infty} \frac{z^2+1}{z-1} = \infty$ . 2

**OPTION-B**

**( For Old Syllabus )**

**( Riemann Integration and Metric Spaces )**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following questions :  $1 \times 10 = 10$

(a) Write the statement of the First Fundamental Theorem of Calculus.

(b) Evaluate  $\int_0^{\infty} e^{-x} dx$ , if it exists.

(c) Prove that  $\Gamma(1) = 1$ .

(d) Define a complete metric space.

(e) Describe an open ball in the discrete metric space  $(X, d)$ .

(f)  $(A \cup B)^0$  need not be  $A^0 \cup B^0$  — Justify it where  $A$  and  $B$  are subsets of a metric space  $(X, d)$ .

(g) Find the derived sets of the intervals  $(0, 1)$  and  $[0, 1]$ .

(h) Let  $A$  and  $B$  be two subsets of a metric space  $(X, d)$ . Which of the following is not correct?

(i)  $A \subseteq B \Rightarrow A' \subseteq B'$

(ii)  $(A \cap B)' \subseteq A' \cap B'$

(iii)  $A' \cap B' \subseteq (A \cap B)'$

(iv)  $(A \cup B)' = A' \cup B'$

(i) The Euclidean metric on  $\mathbb{R}^n$  is defined as

$$(i) \quad d(x, y) = \left\{ \sum_{i=1}^n (x_i - y_i)^2 \right\}^{\frac{1}{2}}$$

$$(ii) \quad d(x, y) = \left\{ \sum_{i=1}^n |x_i - y_i|^p \right\}^{\frac{1}{p}}$$

where  $p \geq 1$

$$(iii) \quad d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

$$(iv) \quad d(x, y) = \sup_{1 \leq i \leq n} |x_i - y_i|$$

where  $x = (x_1, x_2, \dots, x_n)$   
 $y = (y_1, y_2, \dots, y_n)$

are any two points in  $\mathbb{R}^n$ .

(Choose the correct answer)

(j) Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces and  $f: X \rightarrow Y$  be continuous on  $X$ . Then for any  $B \subseteq Y$ .

(i)  $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$

(ii)  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

(iii)  $\overline{f(B)} \subset f(\overline{B})$

(iv)  $f(\overline{B}) \subset \overline{f(B)}$

(Choose the correct answer)

2. Answer the following questions :  $2 \times 5 = 10$

(a) Let  $f(x) = x$  on  $[0, 1]$  and

$$P = \left\{ x_i = \frac{i}{4}, i = 0, 1, \dots, 4 \right\}$$

Find  $L(f, P)$  and  $U(f, P)$ .

(b) Let  $f: [0, a] \rightarrow \mathbb{R}$  be given by

$$f(x) = x^2. \text{ Find}$$

$$\int_0^a f(x) dx$$

(c) Let  $(X, d)$  be a metric space and  $A, B$  be subsets of  $X$ . Prove that  $(A \cap B)^0 = A^0 \cap B^0$ .

(d) If  $A$  is a subset of a metric space  $(X, d)$ , prove that  $d(A) = d(\overline{A})$ .

(e) Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. Prove that if a mapping  $f: X \rightarrow Y$  is continuous on  $X$ , then  $f^{-1}(G)$  is open in  $X$  for all open subsets  $G$  of  $Y$ .

3. Answer **any four** parts :  $5 \times 4 = 20$

(a) Prove that  $f(x) = x^2$  on  $[0, 1]$  is integrable.

(b) Show that  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r}{r^2 + n^2} = \log \sqrt{2}$

(c) Let  $(X, d)$  be a metric space. Define  $d': X \times X \rightarrow \mathbb{R}$  by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$ . Prove that  $d'$  is a metric on  $X$ .

(d) Let  $X = C[a, b]$  and  
 $d(f, g) = \sup\{|f(x) - g(x)| : a \leq x \leq b\}$   
 be the associated metric where  
 $f, g \in X$ . Prove that  $(X, d)$  is a  
 complete metric space.

(e) Let  $(X, d)$  be a metric space. Prove  
 that a finite union of closed sets is  
 closed.  
 Infinite union of closed sets need not  
 to be closed — Justify it. 3+2=5

(f) Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric  
 spaces and  $f : X \rightarrow Y$  be uniformly  
 continuous. If  $\{x_n\}_{n \geq 1}$  is a Cauchy  
 sequence in  $X$ , prove that  $\{f(x_n)\}_{n \geq 1}$   
 is a Cauchy sequence in  $Y$ .

4. Answer **any four** parts : 10×4=40

(a) (i) Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous.  
 Prove that  $f$  is integrable. 5

(ii) Discuss the convergence of the  
 integral  $\int_1^{\infty} \frac{1}{x^p} dx$  for various values  
 at  $p$ . 5

(b) (i) Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous  
 on  $[a, b]$ . Prove that there exists  
 $c \in [a, b]$  such that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

Using it prove that for  $-1 < a < 0$   
 and  $n \in \mathbb{N}$ ,

$$S_n = \int_a^0 \frac{x^n}{1+x} dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

3+2=5

(ii) Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotone.  
 Prove that there exists  $c \in [a, b]$   
 such that

$$\int_a^b f(x) dx = f(a)(c-a) + f(b)(b-c)$$

5

(c) (i) Prove that a convergent sequence  
 in a metric space is a Cauchy  
 sequence.  
 Show that in the discrete metric  
 space every Cauchy sequence is  
 convergent. 3+2=5

(ii) Define an open set in a metric  
 space  $(X, d)$ .  
 Prove that in any metric space  
 $(X, d)$ , each open ball is an open  
 set. 1+4=5

(d) (i) Let  $(X, d)$  be a metric space and  $F$  be a subset of  $X$ . Prove that  $F$  is closed in  $X$  if and only if  $F^c$  is open in  $X$ . 5

(ii) Let  $(X, d)$  be a metric space and  $Y$  a subspace of  $X$ . Let  $Z$  be a subset of  $Y$ . Prove that  $Z$  is open in  $Y$  if and only if there exists an open set  $G \subseteq X$  such that  $Z = G \cap Y$ . 5

(e) (i) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $A \subseteq X$ . Prove that a function  $f : A \rightarrow Y$  is continuous at  $a \in A$  if and only if whenever a sequence  $\{x_n\}$  in  $A$  converges to  $a$ , the sequence  $\{f(x_n)\}$  converges to  $f(a)$ . 6

(ii) Prove that a mapping  $f : X \rightarrow Y$  is continuous on  $X$  if and only if  $f^{-1}(F)$  is closed in  $X$  for all closed subsets  $F$  of  $Y$ . 4

(f) (i) Show that the function  $f : (0, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{x}$  is not uniformly continuous. 5

(ii) Let  $(X, d)$  be a metric space and let  $x \in X$  and  $A \subseteq X$  be non-empty. Prove that  $x \in A$  if and only if  $d(x, A) = 0$ . 5

(g) (i) Define a connected set in a metric space. Prove that if  $Y$  is a connected set in a metric space  $(X, d)$ , then any set  $Z$  such that  $Y \subseteq Z \subseteq \bar{Y}$  is connected. 1+4=5

(ii) Let  $(X, d)$  be a metric space. Prove that the following statements are equivalent :

- (a)  $(X, d)$  is disconnected
- (b) there exists a continuous mapping of  $(X, d)$  onto the discrete two element space  $(X_0, d_0)$ . 5



(h) Let  $(\mathbb{R}, d)$  be the space of real numbers with the usual metric. Prove that a subset  $I$  of  $\mathbb{R}$  is connected if and only if  $I$  is an interval.

