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3 (Sem–1/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : 1×10=10

(a) Find the polar representation of $z = -3i$.

(b) State De Moivre's theorem.

(c) Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex number with $r > 0$ and $t^* \in [0, 2\pi)$. Write down the formula for n distinct n^{th} roots of z_0.

Contd.

(d) Identify the quantifier, set of context and property in the statement, "Every student in this classroom is at least 5*ft* tall."

(e) Define implication. Give an example.

(f) Prove by contradiction "There is no greatest integer".

(g) Let A and B be two sets, write when $A \times B = \phi$. Justify your answer.

(h) What is domain and range for the function $f(x) = \tan x$.

(i) What are the options about the solutions of a system of linear equations ?

(j) Determine h such that the matrix

$$\begin{bmatrix} 2 & 3 & h \\ 6 & 9 & 5 \end{bmatrix}$$ is the augmented matrix of a consistent linear system.

(k) State True **or** False with justification : "Whenever a system has free variables the solution set is infinite."

(l) Write down the system of equations that is equivalent to the vector equation

$$x_1\begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(m) Define Pivot positions in a matrix.

(n) Prove $\vec{U} + \vec{V} = \vec{V} + \vec{U}$ for any $\vec{U}.\vec{V}$ in $\mathbb{R}^n$.

(o) Write the system of equation as a matrix equation

$$3x_1 + x_2 - 5x_3 = 9$$
$$x_2 + 4x_3 = 0$$

(p) Given, $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Compute $x^T A^T$ and $A^T x^T$.

(q) A is an $n \times n$ matrix. Prove statement *(i)* $\Rightarrow$ statement *(ii)*.

(i) A is an invertible matrix

(ii) $\exists a$ $n \times n$ matrix C s.t. $CA = I$

(r) A is an $n \times n$ matrix

Fill in the blank :

If two rows of A are interchanged to produce B, then det $B =$ ________ .

2. Answer ***any five*** : $2 \times 5 = 10$

(a) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Express $z_1 z_2$ in polar form.

(b) Write the 'converse' and 'contrapositive' of the following statement :

"For real numbers x and y, if xy is an irrational number then either x is irrational or y is irrational."

(c) Why may we use the contrapositive of a statement to prove the statement instead of direct proof? Justify using truth table.

(d) Produce counter examples to disapprove the following :

(i) For $x, y \in \mathbb{R}$, $|a| > |b|$ if $a > b$

(ii) For any $x \in \mathbb{R}$, $x^2 \geq x$

(e) Express the empty set as a subset of $\mathbb{R}$ in two different ways.

(f) Express $\mathbb{N}$ as the union of an infinite number of finite sets I_n indexed by $n \in \mathbb{N}$.

(g) Give an example of a relation that is not reflexive, not transitive but is symmetric.

(h) State True **or** False with justification :

An example of a linear combination of vectors $\vec{v}_1$ and $\vec{v}_2$ is $\frac{1}{2}\vec{v}_1$.

(i) Prove that the following vectors are linearly dependent

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

(j) Evaluate the determinant by using row reduction to Echelon form

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

3. Answer ***any four*** : 5×4=20

(a) Compute $z=\left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n$.

(b) Prove that the power set of a set with n elements has 2^n elements. Write down the power set of $S=\{a,b\}$.

(c) Prove that the equivalence classes of an equivalence relation on a set X induces a partition of X.

(d) Prove $(1+x)^n \geq 1+nx$ for $x\in\mathbb{R}$ such that $x>-1$ and for each $n\in N$. Give the name of this inequality.

(e) Balance the chemical equation using vector equation approach the following reaction between potassium permanganate ($KMnO_4$) and manganese sulfate ($MnSO_4$) in water produces manganese dioxide, potassium sulfate and sulfuric acid.

The unbalanced equation is

$$KMnO_4+MnSO_4+H_2O\rightarrow MnO_2+K_2SO_4+H_2SO_4$$

(f) Find the value of h for which the set of vectors is linearly dependent

$$\begin{bmatrix}2\\-4\\1\end{bmatrix},\begin{bmatrix}-6\\7\\-3\end{bmatrix},\begin{bmatrix}8\\h\\4\end{bmatrix}$$

(g) Let A be an $m\times n$ matrix. Prove that the following statements are logically equivalent.

(i) For each $b\in\mathbb{R}^m$, the equation $A\vec{x}=\vec{b}$ has a solution.

(ii) Each $b\in\mathbb{R}^m$ is a linear combination of the columns of A.

(iii) The columns of A span $\mathbb{R}^m$.

(iv) A has a pivot position in every row.

(h) Use Cramer's rule to compute the solutions to the system

$$\begin{aligned}2x_1+x_2\quad\quad &=7\\-3x_1\quad+x_3 &=-8\\x_2+2x_3 &=-3\end{aligned}$$

4. Answer ***any four*** : 10×4=40

(a) *(i)* Prove $\prod_{\substack{1 \le k \le n-1 \\ gcd(k,n)=1}} sin\frac{k\pi}{n} = \frac{1}{2^{\phi(n)}}$ whenever n is not a power of a prime. 5

(ii) Solve the equation

$z^7 - 2iz^4 - iz^3 - 2 = 0$ 5

(b) For any three sets A, B and C, show that

(i) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ 5

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 5

(c) Define graph of a function verify that the set $\{(x, y) \in \mathbb{R}^2 : x = |y|\}$ is not the graph of any function. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c, a \neq 0$. Show that the function is neither one-one nor onto. 2+2+6=10

(d) Let $X = \mathbb{R}$ and let $R = \{(x, y) \in R^2 : xy = 0\}$. When $x \in \mathbb{R}$ is related to $y \in \mathbb{R}$? Define reflexive, symmetric, antisymmetric and transitive relation with examples. 2+2+2+2+2=10

(e) If $A \subseteq N$, what is the least element of A? State and prove Division Algorithm. 2+1+7=10

(f) *(i)* Solve the system : 5

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

(ii) Suppose the system 3

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

is consistent for all possible values of f and g, what can you say about the co-efficients c and d. Justify.

(iii) Suppose a 3×5 co-efficient matrix for a system has three pivot columns. Is the system consistent? Justify. 2

(g) (i) If $\vec{U} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Display $\vec{U}, \vec{V}, \vec{U} - \vec{V}$ using arrows on an xy graph. 3

(ii) List five vectors in the span $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$ 2

(iii) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^4$? Justify. 5

(h) (i) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ 5

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every possible $\vec{b}$ if A is a 3×2 matrix with two pivot positions? 2

(iii) Prove that if a set contains more vectors than the number of entries in each vector then the set is linearly dependent. 3

(i) (i) Define linear transformation. Give an example. 2

(ii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then prove T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$. 3

(iii) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 into $e_2 + 3e_1$. 3

(iv) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$$ 2

(j) (i) Find the inverse of the matrix A (if it exists) by performing suitable row operations on the augmented matrix $[A : I]$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$ 4

(ii) Find the volume of the parallelopiped with one vertex at the origin and adjacent vertices at (1, 0, –2), (1, 2, 4) and (7, 1, 0). 3

(iii) Let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be determined by a 2×2 matrix A. Prove that if S is a parallelogram in $\mathbb{R}^2$ then

$$\{\text{area of } T(S)\} = /\det A/\{\text{area of } S\}$$ 3
