3 (Sem–3/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer **any ten** : 1×10=10

   (i) Write down the formulae of transformation from one pair of rectangular axes to another with same origin.

   (ii) Find the equation to the locus of the point $P(t, 2t)$ if t is a parameter.

(iii) For what value of a, the transformation $x' = -x + 2,$ $y' = ax + 3$ is a translation?

(iv) What is the locus represented by the equation $ax^2 - 5xy + 6y^2 = 0$?

(v) Write down the polar equation of the straight line $x = 0$.

(vi) Under what condition
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
may represents a pair of straight lines?

(vii) What will be the equation of the line $ax + by + c = 0$ if the origin is transferred to the point (α, β)?

(viii) The parabola represented by the equation $y^2 = 4ax$ is not a closed curve. How can you justify it from the given equation?

(ix) Write the relationship between the lengths of semi-major axis, semi-minor axis and the eccentricity for the standard equation of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > a.$$

(x) What conic does the following equation represent?
$$x^2 + 2xy + y^2 - 2x - 1 = 0$$

(xi) What are the direction ratios of the normal to the plane given by equation $ax + by + cz + d = 0$?

(xii) Write down the direction cosines of z-axis.

(xiii) When does the equation
$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$
represent a sphere?

(xiv) When is a plane said to be parallel to a line?

(xv) Mention the condition under which the lines $\dfrac{x - \alpha_1}{l_1} = \dfrac{y - \beta_1}{m_1} = \dfrac{z - \gamma_1}{n_1}$ and

$\dfrac{x - \alpha_2}{l_2} = \dfrac{y - \beta_2}{m_2} = \dfrac{z - \gamma_2}{n_2}$ are coplanar.

(xvi) What are centre and radius of the sphere given by the equation
$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0?$$

(xvii) Define the polar plane of a point (α, β, γ) with respect to the conicoid

$$ax^2 + by^2 + cz^2 = 1.$$

(xviii) What are the coordinates of the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0 ?$$

2. Answer **any five** : $\qquad$ 2×5=10

(a) Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated through an angle $\dfrac{\pi}{3}$.

(b) If $\left(at_1^2, 2at_1\right)$ and $\left(at_2^2, 2at_2\right)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, prove that $t_1t_2 = -1$.

(c) If the two pair of lines

$$x^2 - 2pxy - y^2 = 0 \text{ and}$$

$x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq + 1 = 0$.

(d) If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

(e) Find the equation of the plane containing the lines

$$2x + 3y + 5z - 7 = 0, \quad 3x - 4y + z + 14 = 0$$

and passing through the origin.

(f) Find the equation of the cone whose vertex is at the origin and whose guiding curve is given by

$$x = a, \ y^2 + z^2 = b^2 .$$

(g) Find the equation of the sphere through the circle

$$x^2 + y^2 + z^2 = 9, \ 2x + 3y + 4z = 5$$

and the point (1, 2, 3).

(h) Mention the conditions under which the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents *(i)* a parabola, *(ii)* an ellipse, *(iii)* a hyperbola, and *(iv)* a circle.

(i) Find the perpendicular distance of the point $(1, 4, -2)$ from the plane $2x - 3y + z = 5$.

(j) The axis of a right circular cylinder is

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{2}$$ and its radius is 5.

Find its equation.

3. Answer **any four** :   5×4=20

(a) Prove that the transformation of rectangular axes which converts

$$\frac{X^2}{P} + \frac{Y^2}{Q}$$ into $ax^2 + 2hxy + by^2$ will

convert $\frac{X^2}{P-\lambda} + \frac{Y^2}{Q-\lambda}$ into

$$\frac{ax^2 + 2hxy + by^2 - \lambda(ab - h^2)(x^2 + y^2)}{1 - (a+b)\lambda + (ab - h^2)\lambda^2}.$$

(b) Prove that the equation
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
represents a pair of parallel straight

lines if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(c) Show that the line $lx + my = n$ is a

tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

$$a^2 l^2 + b^2 m^2 = n^2.$$

(d) Prove that the product of the perpendiculars from any point on a hyperbola to the asymptotes is constant.

(e) A plane passes through a fixed point (p, q, r), and cut the axes in A, B, C. Show that the locus of the centre of

the sphere OABC is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$.

(f) Find the centre and radius of the circle
$$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0,$$
$$x - 2y + 2z = 3.$$

(g) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by the lines drawn from O is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

(h) Find the equation of the circular cylinder whose guiding circle is

$$x^2 + y^2 + z^2 = 9, \; x - y + z = 3.$$

4. Answer **any four** : 10×4=40

(a) If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points, where these meet the axes is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c} xy = 0.$$

(b) Obtain the equation of the chord of the conic $\dfrac{l}{r} = 1 + e\cos\theta$, joining the two points on the conic, whose vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$.

(c) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'} + \frac{1}{QS.SQ'} = \text{a constant.}$$

(d) Find the equation of a polar of a given point $P(x_1, y_1)$ with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Hence show that if the polar of a point P with respect to the conic passes through a point Q, then the polar of Q also passes through P.

(e) (i) Show that the locus of the points of intersection of perpendicular tangents to a parabola is its directrix.

(ii) Find the asymptotes of the hyperbola $xy + ax + by = 0$.

(f) What do you mean by skew lines ? How do you define the shortest distance between two such lines ? Find the length and the equations of the line of shortest distance between the lines

$$3x - 9y + 5z = 0, \; x + y - z = 0$$
$$6x + 8y + 3z - 13 = 0, \; x + 2y + z - 3 = 0.$$

(g) Prove that the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2 - b^2} + \frac{z}{c}\sqrt{b^2 - c^2} = \lambda \text{ cuts the}$$

ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$b\sqrt{1 - \frac{\lambda^2}{a^2 - c^2}}.$$

(h) Prove that the lines through (α, β, γ) at right angles to their polars with

respect to $\frac{x^2}{a+b} + \frac{y^2}{2a} + \frac{z^2}{2b} = 1$ generate

the cone

$$(y - \beta)(\alpha z - \gamma x) + (z - \gamma)(\alpha y - \beta x) = 0.$$

What is the peculiarity of the case when $a = b$?

(i) Show that the equation of the cylinder whose generators are parallel to the line

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \text{ and guiding curve is}$$

$x^2 + 2y^2 = 1, z = 3$ is

$3(x^2 + 2y^2 + z^2) + 8yz - 2zx + 6x - 24y - 18z + 24 = 0.$

(j) What do you mean by a director sphere ? Find the equation of the director sphere of the conicoid

$ax^2 + by^2 + cz^2 = 1$. Hence or otherwise prove that the director sphere of the

ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$

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