*Total number of printed pages–16*

**3 (Sem–6/CBCS) MAT HC 1 (N/O)**

**2024**

**MATHEMATICS**

( Honours Core )

Paper : MAT–HC–6016

New Syllabus

***(Riemann Integration and Metric Spaces)***

*Full Marks* : 80

Time : Three hours

Old Syllabus

***( Complex Analysis )***

*Full Marks* : 60

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

*Contd.*

New Syllabus

**(Riemann Integration and Metric Spaces)**

*Full Marks : 80*

Time : Three hours

1. Answer the following as directed : $1\times10=10$

(a) Let $f:[a,b]\to R$ be a bounded function and $P$, $Q$ are partitions of $[a,b]$. If $Q$ is a refinement of $P$, then

(i) $L(f,Q)\le L(f,P)$

(ii) $U(f,P)\le U(f,Q)$

(iii) $U(f)\le L(f)$

(iv) $L(f)\le U(f)$

*(Choose the correct option)*

(b) Find the value of $\int_0^\infty e^{-x}dx$.

(c) Show that $\Gamma(1)=1$.

(d) Define Cauchy sequence in a metric space.

(e) State whether the following statement is true ***or*** false :

"Each subset of a discrete metric space is open."



(f) If the mapping $d:R^2\times R^2\to R$ is defined as $d\left((x_1,y_1),(x_2,y_2)\right)=|x_1-x_2|$, then which one of the following statements is true ?

(i) $d$ is the usual metric on $R^2$

(ii) $d$ is uniform metric on $R^2$

(iii) $d$ is a pseudo metric on $R^2$

(iv) None of the above statements is true

(g) Which of the following statements is not true ?

(i) In a metric space countable union of open sets is open

(ii) In a metric space finite union of closed sets is closed

(iii) A non-empty subset of a metric space is closed if and only if its complement is open

(iv) None of the above statements is true

(h) When is a metric space said to be connected.

(i) State whether the following statement is true **or** false :

"Image of an open set under a continuous function is open."

(j) Under what condition the metric spaces $(X, d_X)$ and $(Y, d_Y)$ are said to be equivalent ?

2. Answer the following questions : 2×5=10

(a) Let $u, v : [a, b] \to R$ be differentiable and $u', v'$ are integrable on $[a, b]$. Then show that

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

(b) Show that a subset $F$ of a metric space $(X, d)$ is closed if and only if $\bar{F} = F$.

(c) Let $(Y, d_Y)$ be a subspace of a metric space $(X, d_X)$ and $S_X(z, r)$ and $S_Y(z, r)$ are open balls with center at $z \in Y$ and radius $r$ in the metric space $(X, d_X)$ and $(Y, d_Y)$ respectively.

Prove that $S_Y(z, r) = S_X(z, r) \cap Y$.



(d) Show that the image of a Cauchy sequence under uniformly continuous function is again a Cauchy sequence.

(e) Show that a contraction mapping on a metric space is uniformly continuous.

3. Answer ***any four*** questions : 5×4=20

(a) Show that a bounded function $f : [a, b] \to R$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

(b) Let $g$ be a continuous function on the closed interval $[a, b]$ and the function $f$ be continuously differentiable on $[a, b]$. Further if $f'$ does not change sign on $[a, b]$, then show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

(c) Let $(X, d)$ be a metric space and the function $d^*: X \times X \to R$ is defined as

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$$

Show that $(X, d^*)$ is a bounded metric space.

(d) Let $Y$ be a non-empty subset of the metric space $(X, d)$. Prove that the subspace $(Y, d_Y)$ is complete if and only if $Y$ is closed on $(X, d)$.

(e) Show that composition of two uniformly continuous functions is also uniformly continuous.

(f) Show that a metric space $(X, d)$ is disconnected if and only if there exists a continuous function of $(X, d)$ onto the discrete two element space $(X_0, d_0)$, i.e., $X_0 = \{0, 1\}$ and $d_0$ is the discrete metric on $X_0$.



4. Answer the following questions : 10×4=40

(a) Let $f$ be a function on an interval $J$ with $n$th derivative $f^{(n)}$ continuous on $J$. If $a, b \in J$, then show that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(b-a)^{n-1} + R_n$$

where, $R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$

***Or***

Let $f:[0, 1] \to R$ be continuous and $c_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $n \in N$.

Then show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(c_i) = \int_0^1 f(x)\,dx.$$

Hence show that

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{r^2 + n^2} = \log\sqrt{2}.$$ 5+5=10



Contd.

(b) Let $l_p$ $(p \geq 1)$ be the set of all sequences of real numbers such that if

$$x = \{x_n\}_{n\geq 1} \in l_p, \text{ then } \sum_{i=1}^{\infty} |x_i|^p < \infty.$$

Prove that the function $d : l_p \times l_p \to R$

defined by $d(x, y) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^p \right\}^{\frac{1}{p}}$

is a metric on $l_p$. Also show that $l_p$ is a complete metric space. 4+6=10

**Or**

(i) Let $(X, d)$ be a metric space and $\{x_n\}_{n\geq !}$, $\{y_n\}_{n\geq !}$ be two sequences in $X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then show that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. 4

(ii) Let $(X, d)$ be a metric space and $Y$ a subspace of $X$. Let $Z$ be a subset of $Y$. Then show that $Z$ is closed in $Y$ if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$. 6



(c) What are meant by contraction mapping and fixed point of a contraction mapping in a metric space? If $T : X \to X$ is a contraction mapping on a complete metric space, then show that $T$ has a unique fixed point. (1+1)+8=10

**Or**

If $(X, d)$ be a metric space, then show that the following statements are equivalent :

(i) $(X, d)$ is disconnected.

(ii) There exist two non-empty disjoint subsets $A$ and $B$, both open in $X$, such that $X = A \cup B$.

(iii) There exist two non-empty disjoint subsets $A$ and $B$, both closed in $X$, such that $X = A \cup B$.

(iv) There exists a proper subset of $X$ that is both open and closed in $X$.

(d) (i) Let $f:[a,b]\to R$ be integrable and $F(x)=\int_a^x f(t)dt$ ; $x\in[a,b]$. Show that $F$ is continuous on $[a,b]$. Also show that $F$ is differentiable at $x\in[a,b]$ if $f$ is continuous at $x\in[a,b]$ and $F'(x)=f(x)$. 7

(ii) Let $(X,d)$ be a metric space and $\rho:X\times X\to R$ be defined by

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)} \quad ; \quad x,y\in X.$$

Show that $d$ and $\rho$ are equivalent metrics. 3

**Or**

(iii) Show that a subset $G$ of a metric space $(X,d)$ is open if and only if it is the union of all open balls contained in $G$. 5

(iv) Give example, with justification, of a homeomorphism from a metric space onto another metric space which is not an isometry. 5

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Old Syllabus

Full Marks : 60

**( Complex Analysis )**

Time : Three hours

1. Answer the following as directed : 1×7=7

(a) Determine the accumulation point of the set $z_n=\frac{i}{n}\,(n=1,2,3,\cdots)$

(b) Describe the domain of $f(z)=\frac{z}{z+\bar{z}}$.

(c) Define an entire function.

(d) Determine the singular points of

$$f(z)=\frac{2z+1}{z\left(z^2+1\right)}$$

(e) The value of loge is

(i) 1

(ii) $1+2n\pi i$

(iii) $2n\pi i$

(iv) 0

(Choose the correct option)

(f) $\lim_{n\to\infty}\left(-2+i\frac{(-1)^n}{n^2}\right)$ is equal to

(i) 0 (ii) $-2$

(iii) $-2+i$ (iv) limit does not exist

*(Choose the correct option)*

(g) The power expression for $\cos z$ is

(i) $\frac{e^z+e^{-z}}{2}$ (ii) $\frac{e^{iz}+e^{-iz}}{2}$

(iii) $\frac{e^{iz}+e^{-iz}}{2i}$ (iv) $\frac{e^z-e^{-z}}{2}$

*(Choose the correct option)*

2. Answer the following questions : 2×4=8

(a) Sketch the set

$$|z-1+i|\le 1$$

(b) Prove that $f'(z)$ exists every where for the function $f(z)=iz+2$.

(c) If $f(z)=\frac{z}{\bar{z}}$, prove that $\lim_{z\to 0} f(z)$ does not exist.

(d) Evaluate $\int_1^2\left(\frac{1}{t}-i\right)^2 dt$.



3. Answer **any three** questions from the following : 5×3=15

(a) (i) Show that if $e^z$ is real, then

$Im\, z = n\pi\ (n=0,\ \pm1,\ \pm2,\cdots)$ 3

(ii) Show that $exp(2\pm 3\pi i)=-e^2$. 2

(b) Suppose that $f(z)=u(x,y)+iv(x,y)$, where $z=x+iy$ and $z_0=x_0+iy_0$, $w_0=u_0+iv_0$. Then prove that

$\lim_{z\to z_0} f(z)=w_0$ if

$\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0$ and

$\lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$

(c) Show that $f'(z)$ exists everywhere, when $f(z)=e^z$.

(d) Evaluate $\int_C \frac{dz}{z}$, where $C$ is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

(e) Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Applying Cauchy's integral formula, evaluate $\int_C \frac{e^{-z} dz}{z - (\pi i/2)}$.

4. Answer **either** (a) **or** (b) and (c):

(a) Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Prove that the first order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$ and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also show that $f'(z) = u_x + iv_x = v_y - iu_y$ where partial derivatives are to be evaluated at $(x_0, y_0)$. 10

**Or**

(b) If $z_0$ and $w_0$ are points in the $z$-plane and $w$-plane respectively, then prove that $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$

Hence show that $\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$

4+2=6



(c) If $w = f(z) = \bar{z}$, examine whether $\frac{dw}{dz}$ exists or not. 4

5. Answer **either** (a) **or** (b):

(a) Let $C$ denote a contour of length $L$ and suppose that a function $f(z)$ is piecewise continuous on $C$. If $M$ is a non-negative constant such that $|f(z)| \leq M$ for all points $z$ on $C$ at which $f(z)$ is defined, then prove that

$$\left| \int_C f(z) dz \right| \leq ML$$

Hence show that

$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3$, where $C$ is the arc of the semicircle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. 10

**Or**

(b) State and prove Liouville's theorem.

6. Answer ***either*** *(a)*, *(b)*, *(c)* **or** *(d)*:

*(a)* Prove that if a series of complex numbers converges, then the $n$th term converges to zero as $n$ tends to infinity. 3

*(b)* Test the convergency of the sequence

$$z_n = \frac{1}{n^3} + i \,(n = 1, 2, \cdots)$$ 3

*(c)* Find Maclaurin's series for the entire function $f(z) = l^z$. 4

***Or***

*(d)* Suppose that a function $f$ is analytic throughout a disc $|z - z_0| < R_0$ centred at $z_0$ and with radius $R_0$. Then prove that $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

where $a_n = \dfrac{f^n(z_0)}{n!} (n = 0, 1, 2, \cdots)$

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